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On the Order of a Restricted System of Equations.[*]

By F. F. Decker.

Section 1.

In an earlier paper[†] the writer gave a proof of the theorem—announced without proof by Salmon[‡]—that the number of solutions of the system of equations arising from the vanishing of all the determinants of the m-th order that can be formed from a matrix with m rows and n columns, $m \not> n$, by the suppression of $n-m$ columns, the elements in the i-th row and j-th column being of degree α_i+a_j in $n-m+1$ non-homogeneous variables is K_{n-m+1} where $K_l=\sum\limits_{i=0}^{i=l} c_i\delta_{l-i}$, c_l representing the sum of all possible products of l different a's and δ_l the sum of all possible products of l α's, repetitions being permissible.

In this paper the system arising from the vanishing of all the determinants of the r-th order that can be formed by the suppression of $n-r$ columns and $m-r$ rows, $r \not> m$, $r \not> n$, is treated. The order is shown to be

$$\begin{vmatrix} K_{n-r+1} & K_{n-r} & \dots\dots K_{n-m+1} \\ K_{n-r+2} & K_{n-r+1} & \dots\dots K_{n-m+2} \\ \dots\dots & \dots\dots & \dots\dots\dots\dots \\ K_{n-2r+m+1} & K_{n-2r+m} & \dots\dots K_{n-r+1} \end{vmatrix} \tag{A}$$

If the degree of every element is b the order reduces to

$$\frac{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_{m+1-i}}{\prod\limits_{i=0}^{i=m-r} {}_{n+m-r}C_i} \cdot b^{(n-r+1)(m-r+1)}, \text{ which may be written}$$

$$\frac{\prod\limits_{i=0}^{i=m-r} {}_{n+i}C_{n-r+1}}{\prod\limits_{i=0}^{i=m-r} {}_{n-r+1+i}C_{n-r+1}} \cdot b^{(n-r+1)(m-r+1)}, \tag{B}$$

a result established by Segre.[§]

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[*] Presented before the American Mathematical Society, April 27, 1918.

[†] "On the Order of a Restricted System of Equations," American Journal of Mathematics, Vol. XXXVII, No. 2, April, 1915.

[‡] Salmon, "Modern Algebra," Fourth Edition, pp. 283–313.

[§] Segre, "Gli ordini delle varietà che annullano dei diversi gradi estratti da una data matrice," *Rendic. R. Accad. Dei Lincei*, Series 5, Vol. IX, session of October 21, 1900.

In § 2 the notation is defined and some preliminary relations are given. Theorem VI simplifies the derivation of the order for $m > n$ after it has been derived for $m \not> n$ (Theorem XII). In § 3 is considered the case of the vanishing of the determinants of two matrices having some common rows. The order of that part of the system for which the determinants of the common rows vanish is found (Theorem IX). In § 4 is established by mathematical induction, with the aid to Theorem V, the relation (A) of § 1 (Theorem XIII), and the relation (B) is established in § 5 (Theorem XIV).

Section 2.

$\begin{Vmatrix} u_{11} \dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots u_{mn} \end{Vmatrix}^{(r)}$ or $\|U_{mn}\|^{(r)}$ or $\begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}^{(r)}$ denotes the aggregate of all determinants of the r-th order that can be formed from the matrix $\begin{Vmatrix} u_{11} \dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots u_{mn} \end{Vmatrix}$ by suppressing $m-r$ rows and $n-r$ columns. That all these determinants vanish will be indicated by

$$\begin{Vmatrix} u_{11} \dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots u_{mn} \end{Vmatrix}^{(r)} = 0 \text{ or } \|U_{mn}\|^{(r)} = 0 \text{ or } \begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}^{(r)} = 0.$$

$M(\bar{h}, \bar{k})$ denotes the matrix formed from $\begin{pmatrix} 1 \dots n \\ 1 \dots m \end{pmatrix}$ by suppressing the first h rows and the last k rows, that is $\begin{pmatrix} 1 \dots n \\ h+1 \dots m-k \end{pmatrix}$.

The order of the system of equations $A=0$ will be indicated by $\bar{A}$. When the (r) is lacking from the symbol it will be understood to be the same as the smaller of the two numbers m, n. If r is one more than the smaller of the numbers m, n, the order symbol is to be understood to represent the number 1, and if r exceeds each of the numbers m, n by more than one, the order symbol is to be understood to represent zero.

ϕ is an operator such the $\phi \bar{M}(\bar{h}_1, \bar{k}_1) \bar{M}(\bar{h}_2, \bar{k}_2) \dots \bar{M}(\bar{h}_s, \bar{k}_s)$ produces the algebraic sum of all the products that can be formed from

$$\bar{M}(\bar{h}_1, \bar{k}_1) \bar{M}(\bar{h}_2, \bar{k}_2) \dots \bar{M}(\bar{h}_s, \bar{k}_s)$$

by interchanging k_1 with each of the other k's in turn, the sign of each product being given by the formula $(-1)^{t_i+1}$ where t_i is the number of inversions of the natural order of the k's in the different permutations, the h's occurring in the natural order. $K_l \equiv \sum_{i=0}^{i=l} c_i \delta_{l-i}$ and $J_l \equiv \sum_{i=0}^{i=l} d_i \gamma_{l-i}$ where $c_l \equiv \sum_1^s \prod_{i=1}^{i=l} a_i$, $\gamma_l \equiv \sum_1^s \prod_{i=1}^{i=l} \alpha_i$, and d and δ differ from c and γ, respectively, only in permitting repetitions of the same letter in forming homogeneous products. $K_0 = J_0 = c_0 = d_0 = \gamma_0 = \delta_0 = 1$. When l is negative the values of the symbols are zero. $K'_s = K_{n-m+s}$.

u_{rs} denotes a function of not less than $(n-r+1)(m-r+1)$ non-homogeneous variables, and, whenever the order is calculated in terms of K's or J's, u_{rs} is considered to be of degree $a_r + a_s$.

Use will be made of the following relations (Theorems I–III), proofs of which may be found in a previous paper on the subject by the writer already referred to.

THEOREM I. $\left(\begin{matrix}\overline{1\dots\dots n}\\ 1\dots\dots m\end{matrix}\right) = K_{n-m+1}$.

THEOREM II. $\left(\begin{matrix}\overline{1\dots\dots n}\\ 1\dots\dots m\end{matrix}\right) = \sum_1^m (-1)^{i+1} \left(\begin{matrix}\overline{i+1\dots\dots n}\\ 1\dots\dots m\end{matrix}\right)^{(m)} \left(\begin{matrix}\overline{i\dots\dots m}\\ 1\dots\dots m\end{matrix}\right)^{(m-i+1)}$

$$= \sum_1^m (-1)^{i+1} \left(\begin{matrix}\overline{1\dots\dots n-i}\\ 1\dots\dots m\end{matrix}\right)^{(m)} \left(\begin{matrix}\overline{n-m+1\dots\dots n-i+1}\\ 1\dots\dots m\end{matrix}\right)^{(m-i+1)}$$

THEOREM III. $\sum_{i=0}^{i=l} (-1)^i K_{l-i} J_i = 0$.

THEOREM IV. $\underset{a=r+1\dots m}{K_n} = K_n - K_{n-1}\sum_1^r \alpha_1 + K_{n-2}\sum_1^r \alpha_1\alpha_2 - \dots\dots + (-1)^r K_{n-r}\sum_1^r \alpha_1\alpha_2\dots\dots\alpha_r$, *where the a's run from* 1 *to n in all the K's, and the α's run from* 1 *to m, except in the case noted.*

To establish this relation we divide the terms of K_n into those that contain α_1 and those that do not. From the definition of K, $K_n = \alpha_1 K_{n-1} + \underset{a=2\dots\dots m}{K_n}$, whence $\underset{a=2\dots\dots m}{K_n} = K_n - \alpha_1 K_{n-1}$. Then we assume that

$$\underset{r\dots\dots m}{K_n} = K_n - K_{n-1}\sum_1^{r-1}\alpha_1 + \dots\dots + (-1)^{l-1} K_{n-l+1}\sum_1^{r-1}\alpha_1\alpha_2\dots\dots\alpha_{l-1}$$
$$+ (-1)^l K_{n-l}\sum_1^{r-1}\alpha_1\alpha_2\dots\dots\alpha_l + \dots\dots$$

It follows that

$$\underset{r+1\dots\dots m}{K_n} = \underset{r\dots\dots m}{K_n} - \alpha_r \underset{r\dots\dots m}{K_{n-1}} = K_n - \left(\alpha_r + \sum_1^{r-1}\alpha_1\right) K_{n-1}$$
$$+ \dots\dots + (-1)^s \left(\alpha_r \sum_1^{r-1}\alpha_1\alpha_2\dots\dots\alpha_{s-1} + \sum_1^{r-1}\alpha_1\alpha_2\dots\dots\alpha_s\right) K_{n-s} + \dots\dots$$
$$= K_n - K_{n-1}\sum_1^r \alpha_1 + \dots\dots + (-1)^s K_{n-s}\sum_1^r \alpha_1\alpha_2\dots\dots\alpha_s$$
$$+ \dots\dots + (-1)^r K_{n-r}\sum_1^r \alpha_1\alpha_2\dots\dots\alpha_r.$$

THEOREM V. $(1-\phi)\begin{pmatrix}\overline{1\ldots\ldots n}\\ s+1\ldots\ldots m\end{pmatrix}\Delta_s=\Delta_{s+1}$, $m \ngtr n$,

$$\text{where } \Delta_l=\begin{vmatrix}\begin{pmatrix}\overline{1\ldots\ldots n}\\ 1\ldots\ldots m-l\end{pmatrix} & \begin{pmatrix}\overline{1\ldots\ldots n}\\ 1\ldots\ldots m-l+1\end{pmatrix} & \ldots\ldots & \begin{pmatrix}\overline{1\ldots\ldots n}\\ 1\ldots\ldots m-1\end{pmatrix}\\ \begin{pmatrix}\overline{1\ldots\ldots n}\\ 2\ldots\ldots m-l\end{pmatrix} & \begin{pmatrix}\overline{1\ldots\ldots n}\\ 2\ldots\ldots m-l+1\end{pmatrix} & \ldots\ldots & \begin{pmatrix}\overline{1\ldots\ldots n}\\ 2\ldots\ldots m-1\end{pmatrix}\\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\\ \begin{pmatrix}\overline{1\ldots\ldots n}\\ l\ldots\ldots m-l\end{pmatrix} & \begin{pmatrix}\overline{1\ldots\ldots n}\\ l\ldots\ldots m-l+1\end{pmatrix} & \ldots\ldots & \begin{pmatrix}\overline{1\ldots\ldots n}\\ l\ldots\ldots m-1\end{pmatrix}\end{vmatrix}$$

$$=\begin{vmatrix}\bar{M}(\bar{0},\bar{l}) & \bar{M}(\bar{0},\overline{l-1}) & \ldots\ldots\bar{M}(\bar{0},\bar{1})\\ \bar{M}(\bar{1},\bar{l}) & \bar{M}(\bar{1},\overline{l-1}) & \ldots\ldots\bar{M}(\bar{1},\bar{1})\\ \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots\\ \bar{M}(\overline{l-1},\bar{l}) & \bar{M}(\overline{l-1},\overline{l-1}) & \ldots\ldots\bar{M}(\overline{l-1},\bar{1})\end{vmatrix}$$

Proof: $\Delta_s=\Sigma(-1)^{t_i+1}\bar{M}(0, k_1)\bar{M}(\bar{1}, \bar{k}_2)\ \ldots\ldots\ \bar{M}(\overline{s-1}, \bar{k}_s)$, where k_1, k_2, ..., k_s is a permutation of the numbers 1, 2, ..., s, and t_i the number of inversions of the natural order of the k's in the different permutations. When ϕ operates on $\bar{M}(\bar{s},\bar{0})\Delta_s$ and the series of first indices is restored to the natural order, all the permutations of the second indices except k_1, k_2, ..., k_s, 0 will be obtained and the number of inversions of the series of second indices will be changed by one. Therefore

$$(1-\phi)\bar{M}(\bar{s},\bar{0})\Delta_s=\Sigma(-1)^{t_i+1}\bar{M}(\bar{0},\bar{k}_1)\bar{M}(\bar{1},\bar{k}_2)\ldots\ldots\bar{M}(\bar{s},\bar{k}_{s+1})=\Delta_{s+1}.$$

THEOREM VI. $\Delta=\Delta'$,*

$$\text{where } \Delta\equiv\begin{vmatrix}K_s & K_{s-1} & \ldots\ldots K_1\ 1\ \ 0\ 0\ldots\ldots 0 & 0\\ K_{s+1} & K_s & \ldots\ldots K_2\ K_1\ 1\ 0\ldots\ldots 0 & 0\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & .\\ K_{t-1} & K_{t-2} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots K_1 & 1\\ K_t & K_{t-1} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots K_2 & K_1\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & ..\\ K_{s+t-1} & K_{s+t-2} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots K_{s+1} & K_s\end{vmatrix} \text{ and } \Delta'\equiv\begin{vmatrix}J_t & J_{t-1} & \ldots\ldots J_{t-s+1}\\ J_{t+1} & J_t & \ldots\ldots J_{t-s+2}\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\\ J_{t+s-1} & J_{t+s-2} & \ldots\ldots J_t\end{vmatrix}$$

and $s \ngtr t$.

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* For the case $m=0$, Theorem VI yields a relation between the total symmetric functions and the elementary products of the a's. If also $s=1$, there results a formula for the total symmetric function H_t in terms of the elementary products, a formula which has already been proved by Roe in the *Transactions of the American Mathematical Society,* Vol. V, No. 2, p. 202, April, 1904.

Proof:

$$\text{et } \Delta^{\text{II}} \equiv \begin{vmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & K_1 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot\cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & 0 & \cdot\cdot & \cdot & \dots & K_1 & 1 \\ (-1)^{t-1}J_t & (-1)^{t-2}J_{t-1} & \dots & (-1)^{t-s}J_{t-s+1} & \cdot\cdot & \cdot & \dots & K_2 & K_1 \\ (-1)^{t-1}(K_1J_t-J_{t+1}) & (-1)^{t-2}(K_1J_{t-1}-J_t) & \dots & (-1)^{t-s}(K_1J_{t-s+1}-J_{t-s+2}) & \cdot\cdot & \cdot & \dots & K_3 & K_2 \\ \dots\dots & \dots\dots & \dots & \dots\dots & & \cdot & \dots & & \cdot\cdot \\ \sum\limits_{t}^{s+t-1}(-1)^{i-1}K_{s+t-1-i}J_i & \sum\limits_{t-1}^{s+t-2}(-1)^{i-1}K_{s+t-2-i}J_i & \dots & \sum\limits_{t-s+1}^{t}(-1)^{i-1}K_{t-i}J_i & K_{t-1} & K_{t-2} & \dots & K_{s+1} & K_s \end{vmatrix}$$

$$\Delta^{\text{III}} \equiv \begin{vmatrix} (-1)^{t-1}J_t & (-1)^{t-2}J_{t-1} & \dots & (-1)^{t-s}J_{t-s+1} \\ (-1)^{t-1}(K_1J_t-J_{t+1}) & (-1)^{t-2}(K_1J_{t-1}-J_t) & \dots & (-1)^{t-s}(K_1J_{t-s+1}-J_{t-s+2}) \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ \sum\limits_{t}^{s+t-1}(-1)^{i-1}K_{s+t-1-i}J_i & \sum\limits_{t-1}^{s+t-2}(-1)^{i-1}K_{s+t-2-i}J_i & \dots & \sum\limits_{t-s+1}^{t}(-1)^{i-1}K_{t-i}J_i \end{vmatrix},$$

$$\Delta^{\text{IV}} \equiv \begin{vmatrix} +J_t & +J_{t-1} & \dots & +J_{t-s+1} \\ -J_{t+1} & -J_t & \dots & -J_{t-s+2} \\ +J_{t+2} & +J_{t+1} & \dots & +J_{t-s+3} \\ \dots & \dots & \dots & \dots \\ (-1)^{s-1}J_{t+s-1} & (-1)^{s-1}J_{t+s-2} & \dots & (-1)^{s-1}J_t \end{vmatrix} \text{ and } D \equiv \begin{vmatrix} 1 & 0 & \dots & 0 \\ K_1 & 1 & \dots & 0 \\ \cdot\cdot & \cdot & \dots & \cdot \\ K_{s-1} & K_{s-2} & \dots & 1 \end{vmatrix} \equiv 1.$$

also let $l=(t-s)s$,

$$p=l+\frac{s}{2}(2t-s-1), \quad q=\left\{\begin{array}{l} p+\dfrac{s}{2}, \text{ when } s \text{ is even}, \\ p+\dfrac{s-1}{2}, \text{ when } s \text{ is odd}, \end{array}\right\}, \ q \text{ therefore}$$

always being even.

If for the K's of the first column of Δ the values given by Theorem III are substituted and the result expressed as the algebraic sum of $s+t-1$ determinants by using the corresponding terms of the elements of the first column for the first columns, then the process repeated for the second, third, ..., s-th column in turn, the sum of the determinants that do not vanish on account of columns differing by a common factor is identical with Δ^{II}. If Δ^{II} is expanded by Laplace's development in terms of its first $t-s$ rows, the result is $(-1)^l \cdot \Delta^{\text{III}}$. After the factor $(-1)^{t-1}$ is removed from the first column, $(-1)^{t-2}$ from the second, and so on, the resulting determinant is, according to the product theorem for determinants, $D \cdot \Delta^{\text{IV}}$. Thus

$$\Delta=(-1)^p \cdot D \cdot \Delta^{\text{IV}}=(-1)^q \cdot \Delta'=\Delta'.$$

Section 3.

$\binom{\overline{1\ldots\ldots n}}{1\ldots\ldots m}^{(m-1)}$ will now be considered. $\binom{1\ldots\ldots n}{1\ldots\ldots m}^{(m-1)} \neq 0$ unless $\binom{1\ldots\ldots n}{1\ldots\ldots m-1}=0$ and $\binom{1\ldots\ldots n}{2\ldots\ldots m}=0$. It vanishes with them except when $\binom{1\ldots\ldots n}{2\ldots\ldots m-1}=0$.[*] Let $\binom{\overline{1\ldots\ldots n}}{1\ldots\ldots m}^{(m-1)}=\binom{\overline{1\ldots\ldots n}}{1\ldots\ldots m-1}\cdot\binom{\overline{1\ldots\ldots n}}{2\ldots\ldots m}-\overline{X}$. The value of $\overline{X}$ will now be found, or rather a quantity satisfying a more general condition will be found (Theorem IX), and from it the value of $\overline{X}$ will follow as a special case. First, however, some preliminary theorems (VII and VIII) will be proved.

THEOREM VIIa. $\binom{\overline{1\ldots\ldots 6}}{1\ldots\ldots 5}=\binom{\overline{123}}{12}+\binom{\overline{12}}{12}\cdot\binom{\overline{456}}{345}+\binom{\overline{3456}}{345}$.

This relation will be proved by a system of sections. The section of $\binom{1\ldots\ldots 6}{1\ldots\ldots 5}=0$ by the spreads $\binom{1}{2\ldots\ldots 5}=0$ degenerates into two parts of the same dimension for one of which, say A, $\binom{1}{1}$ and $\binom{2\ldots\ldots 6}{1\ldots\ldots 5}$ also vanish, and for the other $B\equiv\binom{2\ldots\ldots 6}{2\ldots\ldots 5}=0$. Thus $\binom{\overline{1\ldots\ldots 6}}{1\ldots\ldots 5}=\overline{A}+\overline{B}$. Again, taking the section of $A=0$ by $\binom{23}{345}^{(1)}=0$, the section of $B=0$ by $\binom{2}{345}=0$, and then taking the section of one of the parts into which $B=0$ degenerates by $\binom{3}{345}=0$, it is found that

$$\binom{\overline{1\ldots\ldots 6}}{1\ldots\ldots 5}=\binom{\overline{1}}{1}\left[\binom{\overline{23}}{12}+\binom{\overline{456}}{345}\right]+\binom{\overline{2}}{2}\left[\binom{\overline{3}}{2}+\binom{\overline{456}}{345}\right]+\binom{\overline{3\ldots\ldots 6}}{345}$$

$$=\overline{C}+\overline{D}+\overline{E},\text{ where }\overline{C}=\binom{\overline{1}}{1}\binom{\overline{23}}{12}+\binom{\overline{23}}{2},$$

$$\overline{D}=\left[\binom{\overline{1}}{1}+\binom{\overline{2}}{2}\right]\binom{\overline{456}}{345}\text{ and }\overline{E}=\binom{\overline{3\ldots\ldots 6}}{345}.$$

Now $\overline{C}$ is seen to give the first term required and $\overline{D}$ the second, by taking sections by $\binom{1}{2}=0$. The theorem follows.

THEOREM VII. $\binom{\overline{1\ldots\ldots n}}{1\ldots\ldots m}=\sum_{i=n-m+k}^{i=k-1}\binom{\overline{1\ldots\ldots i}}{1\ldots\ldots k}^{(k)}\binom{\overline{i+2\ldots\ldots n}}{k+1\ldots\ldots m}^{(m-k)}$

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[*] **See author's paper already referred to.**

Proof: Theorem VIIa shows the relation to be valid for $n=6$, $m=5$, $k=2$. It will be assumed to be valid for $n-1$, m, k and for $n-1$, $m-1$, $k-1$. Then taking the section of $\begin{pmatrix}1\ldots n\\1\ldots m\end{pmatrix}=0$ by $\begin{pmatrix}1\\2\ldots m\end{pmatrix}=0$, it follows that

$$\begin{aligned}\begin{pmatrix}\overline{1\ldots n}\\1\ldots m\end{pmatrix}&=\begin{pmatrix}\overline{1}\\1\end{pmatrix}\begin{pmatrix}\overline{2\ldots n}\\1\ldots m\end{pmatrix}+\begin{pmatrix}\overline{2\ldots n}\\2\ldots m\end{pmatrix}\\&=\begin{pmatrix}\overline{1}\\1\end{pmatrix}\cdot\sum_{i=n-m+k}^{i=k-1}\begin{pmatrix}\overline{2\ldots i}\\1\ldots k\end{pmatrix}^{(k)}\begin{pmatrix}\overline{i+2\ldots n}\\k+1\ldots m\end{pmatrix}^{(m-k)}\\&\qquad+\sum_{i=n-m+k}^{i=k-1}\begin{pmatrix}\overline{2\ldots i}\\2\ldots k\end{pmatrix}^{(k-1)}\begin{pmatrix}\overline{i+2\ldots n}\\k+1\ldots m\end{pmatrix}^{(m-k)}\end{aligned}$$

But $\begin{pmatrix}\overline{1}\\1\end{pmatrix}\begin{pmatrix}\overline{2\ldots i}\\1\ldots k\end{pmatrix}^{(k)}+\begin{pmatrix}\overline{2\ldots i}\\2\ldots k\end{pmatrix}^{(k-1)}=\begin{pmatrix}\overline{1\ldots i}\\1\ldots k\end{pmatrix}^{(k)}$, as may be seen by taking the section of the manifold $\begin{pmatrix}1\ldots i\\1\ldots k\end{pmatrix}=0$ by $\begin{pmatrix}1\\2\ldots k\end{pmatrix}=0$, and the theorem follows.

Consider the two systems of equations

$$I\equiv\begin{cases}\begin{pmatrix}1\ldots k\\1\ldots s\end{pmatrix}=0\\\begin{pmatrix}j\ldots n\\1\ldots s\end{pmatrix}^{(s)}=0\end{cases},\quad II\equiv\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}=0,\qquad s\ngtr j\ngtr k\ngtr n.$$

The vanishing of I is a necessary but not sufficient condition for the vanishing of II. For the system I there may be obtained an equivalent system III by means of a formula previously found by the writer—the formula from which $\begin{pmatrix}\overline{1\ldots n}\\1\ldots m\end{pmatrix}$ is found in Theorem II—and, in general, for a part of this system II vanishes and for a part it does not. Let $Y_{j,k}$ represent the part of the system III, and therefore the part of the system I, for which II vanishes. The value of $\overline{Y}_{j,k}$ will now be investigated.

THEOREM VIIIa. *If* $j>k-s+2$, $\overline{Y}_{j,k}=0$.

For in this case $\begin{pmatrix}1\ldots k\\1\ldots s\end{pmatrix}$ and $\begin{pmatrix}j\ldots n\\1\ldots s\end{pmatrix}^{(s)}$ combined contain not more than $[k-s+1]+[n-j+1)-s+1]$ linearly independent determinants, which is less than $n-s+1$, the number contained in $\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}$ and $\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}$ can not, in general, be made to vanish with $\begin{pmatrix}1\ldots k\\1\ldots s\end{pmatrix}$ and $\begin{pmatrix}j\ldots n\\1\ldots s\end{pmatrix}^{(s)}$.

THEOREM VIIIb. *If* $j=k-s+2$ *and* $k=n-1$, *then* $\overline{Y}_{j,\,k}=\overline{Y}_{n-s+1,\,n-1}$ $=\overline{\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}}$.

Proof: $\overline{\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}}=\sum\limits_1^s(-1)^{i+1}\overline{\begin{pmatrix}1\ldots n-i\\1\ldots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}n-s+1\ldots n-i+1\\1\ldots s\end{pmatrix}}^{(s-i+1)}$, by Theorem II. Therefore $\overline{\begin{pmatrix}1\ldots n-1\\1\ldots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}n-s+1\ldots n\\1\ldots s\end{pmatrix}}^{(s)}=\overline{\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}}^{(s)}+\overline{F}$, where none of the matrices in F contain the n-th column and for none of them, therefore, does $\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}$ vanish. Therefore $\overline{Y}_{n-s+1,\,n-1}=\overline{\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}}^{(s)}$.

THEOREM VIIIc. *If* $j=k-s+2$, $\overline{Y}_{j,\,k}=\overline{Y}_{k-s+2,\,k}=\overline{\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}}$.

Proof: By using the value of $\overline{\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}}$ of Theorem VIIIb and reducing $\overline{\begin{pmatrix}1\ldots n-1\\1\ldots s\end{pmatrix}}$ by means of Theorem II

$$\begin{aligned}\overline{\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}}&=\overline{\begin{pmatrix}n-s+1\ldots n\\1\ldots s\end{pmatrix}}\Bigg[\sum_1^s(-1)^{i+1}\overline{\begin{pmatrix}1\ldots n-i-1\\1\ldots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}n-s\ldots n-i\\1\ldots s\end{pmatrix}}^{(s-i+1)}\Bigg]\\&\quad+\sum_1^{s-1}(-1)^i\overline{\begin{pmatrix}1\ldots n-i-1\\1\ldots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}n-s+1\ldots n-i\\1\ldots s\end{pmatrix}}^{(s-i)}\\&=\overline{\begin{pmatrix}1\ldots n-2\\1\ldots s\end{pmatrix}}^{(s)}\Bigg[\overline{\begin{pmatrix}n-s\ldots n-1\\1\ldots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}n-s+1\ldots n\\1\ldots s\end{pmatrix}}^{(s)}-\overline{\begin{pmatrix}n-s+1\ldots n-1\\1\ldots s\end{pmatrix}}^{(s-1)}\Bigg]\\&\quad+\overline{\begin{pmatrix}1\ldots n-3\\1\ldots s\end{pmatrix}}^{(s)}\Bigg[-\overline{\begin{pmatrix}n-s\ldots n-2\\1\ldots s\end{pmatrix}}^{(s-1)}\overline{\begin{pmatrix}n-s+1\ldots n\\1\ldots s\end{pmatrix}}^{(s)}\\&\quad+\overline{\begin{pmatrix}n-s+1\ldots n-2\\1\ldots s\end{pmatrix}}^{(s-2)}\Bigg]+\overline{\begin{pmatrix}1\ldots n-4\\1\ldots s\end{pmatrix}}^{(s)}\Big[\;\Big]+\ldots\\&=\overline{\begin{pmatrix}1\ldots n-2\\1\ldots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}n-s\ldots n\\1\ldots s\end{pmatrix}}^{(s)}+\overline{\begin{pmatrix}1\ldots n-3\\1\ldots s\end{pmatrix}}^{(s)}\Bigg[\overline{\begin{pmatrix}n-s-1\ldots n\\1\ldots s\end{pmatrix}}^{(s)}\\&\quad-\overline{\begin{pmatrix}n-s-1\ldots n-2\\1\ldots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}n-s\ldots n\\1\ldots s\end{pmatrix}}^{(s)}\Bigg]+\overline{\begin{pmatrix}1\ldots n-4\\1\ldots s\end{pmatrix}}^{(s)}\Big[\;\Big]+\ldots.\end{aligned}$$

We will suppose this relation to be valid in the formula obtained for $\overline{\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}}^{(s)}$ after $n-k-2$ reductions have been made, that is, that

$$\begin{aligned}\overline{\begin{pmatrix}1\ldots n\\1\ldots s\end{pmatrix}}&=\overline{\begin{pmatrix}1\ldots k+1\\1\ldots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}k-s+3\ldots n\\1\ldots s\end{pmatrix}}^{(s)}+\overline{\begin{pmatrix}1\ldots k\\1\ldots s\end{pmatrix}}^{(s)}\Bigg[\overline{\begin{pmatrix}k-s+2\ldots n\\1\ldots s\end{pmatrix}}^{(s)}\\&\quad-\overline{\begin{pmatrix}k-s+2\ldots k+1\\1\ldots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}k-s+3\ldots n\\1\ldots s\end{pmatrix}}^{(s)}\Bigg]+\overline{\begin{pmatrix}1\ldots k-1\\1\ldots s\end{pmatrix}}^{(s)}\Big[\;\Big]+\ldots.\end{aligned}$$

Reducing $\begin{pmatrix}\overline{1\ldots.k+1}\\1\ldots.s\end{pmatrix}^{(s)}$ by Theorem II,

$$\begin{aligned}\begin{pmatrix}\overline{1\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}=\begin{pmatrix}\overline{k-s+3\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}&\Bigg[\begin{pmatrix}\overline{1\ldots.k}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots.k+1}\\1\ldots.s\end{pmatrix}^{(s)}\\&-\begin{pmatrix}\overline{1\ldots.k-1}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots.k}\\1\ldots.s\end{pmatrix}^{(s-1)}+\ldots.\Bigg]\\&+\begin{pmatrix}\overline{1\ldots.k}\\1\ldots.s\end{pmatrix}^{(s)}\Bigg[\begin{pmatrix}\overline{k-s+2\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}\\&-\begin{pmatrix}\overline{k-s+2\ldots.k+1}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+3\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}\Bigg]+\ldots.\\&=\begin{pmatrix}\overline{1\ldots.k}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}+\begin{pmatrix}\overline{1\ldots.k-1}\\1\ldots.s\end{pmatrix}^{(s)}\Big[\;\Big]+\ldots.\,.\end{aligned}$$

Hence $\begin{pmatrix}\overline{1\ldots.k}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}=\begin{pmatrix}\overline{1\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}+\overline{F}$, where none of the matrices in F contain the n-th column. Therefore $Y_{j,k}=\overline{Y}_{k-s+2,k}=\begin{pmatrix}\overline{1\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}$.

THEOREM VIIId. *If* $j=k-s+1$,

$$\overline{Y}_{j,k}=\overline{Y}_{k-s+1,k}=\begin{pmatrix}\overline{k-s+1\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{1\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}.$$

Proof:

$$\begin{aligned}\begin{pmatrix}\overline{1\ldots.k}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+1\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}=\begin{pmatrix}\overline{1\ldots.k}\\1\ldots.s\end{pmatrix}&\Bigg[\begin{pmatrix}\overline{k-s+2\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+1\ldots.k}\\1\ldots.s\end{pmatrix}^{(s)}\\&-\begin{pmatrix}\overline{k-s+3\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+2\ldots.k}\\1\ldots.s\end{pmatrix}^{(s-1)}\\&+\begin{pmatrix}\overline{k-s+4\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{k-s+3\ldots.k}\\1\ldots.s\end{pmatrix}^{(s-2)}-\ldots.\Bigg].\end{aligned}$$

Therefore

$$\begin{aligned}Y_{k-s+1,k}=&\begin{pmatrix}\overline{k-s+1\ldots.k}\\1\ldots.s\end{pmatrix}^{(s)}\overline{Y}_{k-s+2,k}\\&-\begin{pmatrix}\overline{k-s+2\ldots.k}\\1\ldots.s\end{pmatrix}^{(s-1)}\overline{Y}_{k-s+3,k}+\begin{pmatrix}\overline{k-s+3\ldots.k}\\1\ldots.s\end{pmatrix}^{(s-2)}\overline{Y}_{k-s+4,k}-\ldots.\,.\end{aligned}$$

But $\overline{Y}_{k-s+2,k}=\begin{pmatrix}\overline{1\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}$ by Theorem VIIIc and $\overline{Y}_{k-s+3,k}=\overline{Y}_{k-s+4,k}=\ldots.=0$ by Theorem VIIIa. Therefore

$$\overline{Y}_{j,k}=\overline{Y}_{k-s+1,k}=\begin{pmatrix}\overline{k-s+1\ldots.k}\\1\ldots.s\end{pmatrix}^{(s)}\begin{pmatrix}\overline{1\ldots.n}\\1\ldots.s\end{pmatrix}^{(s)}.$$

THEOREM VIII. *The order* $\overline{Y}_{j,k}$ *of the part of the system* $\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}=0$, $\begin{pmatrix}j\dots n\\1\dots s\end{pmatrix}^{(s)}=0$ *for which* $\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}^{(s)}=0$ *is*

$$\overline{\begin{pmatrix}j\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)},\quad s\ngtr j\ngtr k\ngtr n.$$

Proof: Suppose it was already shown that

$$\overline{Y}_{j+i,k}=\overline{\begin{pmatrix}j+i\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)},\quad i=1\dots k-j.$$

Since $\overline{\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}j\dots n\\1\dots s\end{pmatrix}}^{(s)}$

$$=\overline{\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}}^{(s)}\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j+i\dots n\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)},$$

by Theorem II,

$$=\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)}\left[\overline{\begin{pmatrix}1\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}j+i\dots n\\1\dots s\end{pmatrix}}^{(s)}\right].$$

Therefore

$$\overline{Y}_{j,k}=\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)}\overline{Y}_{j+i,k}$$

$$=\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)}\overline{\begin{pmatrix}j+i\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)}$$

$$=\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)}\sum_{i=1}^{i=s}(-1)^{i+1}\overline{\begin{pmatrix}j+i\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}j-1+i\dots j+s-1\\1\dots s\end{pmatrix}}^{(s-i+1)}$$

$$=\overline{\begin{pmatrix}j\dots k\\1\dots s\end{pmatrix}}^{(s)}\overline{\begin{pmatrix}1\dots n\\1\dots s\end{pmatrix}}^{(s)},\ \text{by Theorem II.}$$

It is to be noticed that Theorems VIIIa, b, c are special cases of Theorem VIII, since by definition of the order symbol $\overline{\begin{pmatrix}j\dots k\\1\dots s\end{pmatrix}}^{(s)}$ has the value 1 when $j=k-s+2$, and the value 0 when $j>k-s+2$.

THEOREM IX. *The order* $\overline{X}_{h,l}$ *of the part of the system* $\begin{pmatrix}1\dots n\\1\dots l\end{pmatrix}^{(l)}=0$, $\begin{pmatrix}1\dots n\\h\dots m\end{pmatrix}^{(m-h+1)}=0$, *for which* $\begin{pmatrix}1\dots n\\h\dots l\end{pmatrix}^{(l-h+1)}=0$ *is*

$$\overline{\begin{pmatrix}1\dots n\\h\dots l\end{pmatrix}}^{(l-h+1)}\overline{\begin{pmatrix}1\dots n\\1\dots m\end{pmatrix}}^{(m)},\quad h\ngtr l\ngtr m\ngtr n.$$

Proof: $\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots l\end{pmatrix}^{(l)}=\sum_{i=n-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots\ldots i}\\1\ldots\ldots h-1\end{pmatrix}^{(h-1)}\begin{pmatrix}\overline{i+2\ldots\ldots n}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}$

and $\begin{pmatrix}\overline{1\ldots\ldots n}\\h\ldots\ldots m\end{pmatrix}^{(m-h+1)}=\sum_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{1\ldots\ldots j}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots\ldots n}\\l+1\ldots\ldots m\end{pmatrix}^{(m-l)}$,

according to Theorem VII. Therefore

$$\begin{aligned}&\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots l\end{pmatrix}^{(l)}\begin{pmatrix}\overline{1\ldots\ldots n}\\h\ldots\ldots m\end{pmatrix}^{(m-h+1)}\\&=\sum_{i=n-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots\ldots i}\\1\ldots\ldots h-1\end{pmatrix}^{(h-1)}\begin{pmatrix}\overline{i+2\ldots\ldots n}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\sum_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{1\ldots\ldots j}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots\ldots n}\\l+1\ldots\ldots m\end{pmatrix}^{(m-l)}\\&=\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots\ldots i}\\1\ldots\ldots h-1\end{pmatrix}^{(h-1)}\cdot\sum_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{1\ldots\ldots j}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{i+2\ldots\ldots n}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots\ldots n}\\l+1\ldots\ldots m\end{pmatrix}^{(m-l)}.\end{aligned}$$

Therefore

$$\begin{aligned}\bar{X}_{h,l}&=\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots\ldots i}\\1\ldots\ldots h-1\end{pmatrix}^{(h-1)}\cdot\sum_{j=n-m+l}^{j=l-h}\bar{Y}_{i+2,j}\begin{pmatrix}\overline{j+2\ldots\ldots n}\\l+1\ldots\ldots m\end{pmatrix}^{(m-l)}\\&=\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots\ldots i}\\1\ldots\ldots h-1\end{pmatrix}^{(h-1)}\cdot\sum_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{i+2\ldots\ldots j}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{1\ldots\ldots n}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots\ldots n}\\l+1\ldots\ldots m\end{pmatrix}^{(m-l)},\end{aligned}$$

by Theorem VIII,

$$\begin{aligned}&=\begin{pmatrix}\overline{1\ldots\ldots n}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots\ldots i}\\1\ldots\ldots h-1\end{pmatrix}^{(h-1)}\cdot\sum_{j=n-m+l}^{j=l-h}\begin{pmatrix}\overline{i+2\ldots\ldots j}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{j+2\ldots\ldots n}\\l+1\ldots\ldots m\end{pmatrix}^{(m-l)}\\&=\begin{pmatrix}\overline{1\ldots\ldots n}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\sum_{i=j-l+h-1}^{i=h-2}\begin{pmatrix}\overline{1\ldots\ldots i}\\1\ldots\ldots h-1\end{pmatrix}^{(h-1)}\begin{pmatrix}\overline{i+2\ldots\ldots n}\\h\ldots\ldots m\end{pmatrix}^{(m-h+1)},\text{ by Theorem VII,}\\&=\begin{pmatrix}\overline{1\ldots\ldots n}\\h\ldots\ldots l\end{pmatrix}^{(l-h+1)}\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots m\end{pmatrix}^{(m)},\text{ by Theorem VII.}\end{aligned}$$

COROLLARY: $\bar{X}_{h,k}=\phi\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots l\end{pmatrix}\begin{pmatrix}\overline{1\ldots\ldots n}\\h\ldots\ldots m\end{pmatrix}$, *according to the definition of* ϕ.

Section 4.

THEOREM X. *If* $m \not> n$, $\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots m\end{pmatrix}^{(m-1)}=\begin{vmatrix}\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots m-1\end{pmatrix}&\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots m\end{pmatrix}\\\begin{pmatrix}\overline{1\ldots\ldots n}\\2\ldots\ldots m-1\end{pmatrix}&\begin{pmatrix}\overline{1\ldots\ldots n}\\2\ldots\ldots m\end{pmatrix}\end{vmatrix}$.

Proof: As previously pointed out $\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots m\end{pmatrix}^{(m-1)}=\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots m-1\end{pmatrix}\begin{pmatrix}\overline{1\ldots\ldots n}\\2\ldots\ldots m\end{pmatrix}$

$-\bar{X}_{2,m-1}=\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots m-1\end{pmatrix}\begin{pmatrix}\overline{1\ldots\ldots n}\\2\ldots\ldots m\end{pmatrix}-\begin{pmatrix}\overline{1\ldots\ldots n}\\1\ldots\ldots m\end{pmatrix}\begin{pmatrix}\overline{1\ldots\ldots n}\\2\ldots\ldots m-1\end{pmatrix}$, by Theorem IX.

The theorem follows.

COROLLARY: $\begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.\mathrm{m}\end{pmatrix}^{(m-1)}=\begin{vmatrix}K'_2 & K'_1\\ K'_3 & K'_2\end{vmatrix}$, *where* $K'_i=K_{n-m+i}$, $m \ngtr n$.

Proof: $\begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m\end{pmatrix}^{(m-1)}=\begin{vmatrix}\underset{a=1\ldots.m-1}{K'_2} & \underset{a=1\ldots.m}{K'_1}\\ \underset{a=2\ldots.m-1}{K'_3} & \underset{a=2\ldots.m}{K'_2}\end{vmatrix}$, by theorems X and I,

$=\begin{vmatrix}\underset{a=1\ldots.m}{K'_2-\alpha_m K'_1} & \underset{a=1\ldots.m}{K'_1}\\ \underset{a=2\ldots.m}{K'_3-\alpha_m K'_2} & \underset{a=2\ldots.m}{K'_2}\end{vmatrix}$, by Theorem IV,

$$=\begin{vmatrix}\underset{a=1\ldots.m}{K'_2} & \underset{a=1\ldots.m}{K'_1}\\ \underset{a=2\ldots.m}{K'_3} & \underset{a=2\ldots.m}{K'_2}\end{vmatrix}.$$

By introducing α_1 into the second row in a similar way, the corollary follows.

THEOREM XI. *If* $m \ngtr n \begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m\end{pmatrix}^{(m-2)}$

$$=\begin{vmatrix}\begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m-2\end{pmatrix} & \begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m-1\end{pmatrix} & \begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m\end{pmatrix}\\ \begin{pmatrix}\overline{1\ldots.n}\\ 2\ldots.m-2\end{pmatrix} & \begin{pmatrix}\overline{1\ldots.n}\\ 2\ldots.m-1\end{pmatrix} & \begin{pmatrix}\overline{1\ldots.n}\\ 2\ldots.m\end{pmatrix}\\ \begin{pmatrix}1\ldots.n\\ 3\ldots.m-2\end{pmatrix} & \begin{pmatrix}1\ldots.n\\ 3\ldots.m-1\end{pmatrix} & \begin{pmatrix}1\ldots.n\\ 3\ldots.m\end{pmatrix}\end{vmatrix}= \textit{say}\ \Delta_3.$$

Proof: $\begin{pmatrix}1\ldots.n\\ 1\ldots.m\end{pmatrix}^{(m-2)} \neq 0$ unless $\begin{pmatrix}1\ldots.n\\ 1\ldots.m-1\end{pmatrix}^{(m-2)}=0$ and $\begin{pmatrix}1\ldots.n\\ 3\ldots.m\end{pmatrix}=0$. It vanishes with them except when $\begin{pmatrix}1\ldots.n\\ 3\ldots.m-1\end{pmatrix}=0.$*

Hence $\begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m\end{pmatrix}^{(m-2)}=\begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m-1\end{pmatrix}^{(m-2)}\cdot\begin{pmatrix}\overline{1\ldots.n}\\ 3\ldots.m\end{pmatrix}-\bar{X}_{3,\,m-1}$

$=(1-\phi)\begin{pmatrix}\overline{1\ldots.n}\\ 3\ldots.m\end{pmatrix}\begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m-1\end{pmatrix}^{(m-2)}$,

by Theorem IX, corollary,

$=(1-\phi)\begin{pmatrix}\overline{1\ldots.n}\\ 3\ldots.m\end{pmatrix}\begin{vmatrix}\begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m-1\end{pmatrix} & \begin{pmatrix}\overline{1\ldots.n}\\ 1\ldots.m\end{pmatrix}\\ \begin{pmatrix}\overline{1\ldots.n}\\ 2\ldots.m-1\end{pmatrix} & \begin{pmatrix}\overline{1\ldots.n}\\ 2\ldots.m\end{pmatrix}\end{vmatrix}$,

by Theorem X,

$=\Delta_3$, by Theorem V.

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* See author's paper already referred to.

COROLLARY: $\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m\end{matrix}\right)^{(m-2)}=\begin{vmatrix}K'_3 & K'_2 & K'_1\\ K'_4 & K'_3 & K'_2\\ K'_5 & K'_4 & K'_3\end{vmatrix}$, $K'_i=K_{n-m+i}$, $m \ngtr n$.

Proof: $\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m\end{matrix}\right)^{(m-2)}=\begin{vmatrix}\underset{a=1\ldots m-2}{K'_3} & \underset{a=1\ldots m-1}{K'_2} & \underset{a=1\ldots m}{K'_1}\\ \underset{a=2\ldots m-2}{K'_4} & \underset{a=2\ldots m-1}{K'_3} & \underset{a=2\ldots m}{K'_2}\\ \underset{a=3\ldots m-2}{K'_5} & \underset{a=3\ldots m-1}{K'_4} & \underset{a=3\ldots m}{K'_3}\end{vmatrix}$, by Theorems XI and I.

By means of Theorem IV the missing α's may be introduced as in the proof of the corollary of Theorem X, completing the proof.

THEOREM XII. *If* $m \ngtr n$,

$$\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m\end{matrix}\right)^{(r)}=\begin{vmatrix}\left(\begin{matrix}\overline{1\ldots n}\\1\ldots r\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\1\ldots r+1\end{matrix}\right) & \cdots\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m\end{matrix}\right)\\ \left(\begin{matrix}\overline{1\ldots n}\\2\ldots r\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\2\ldots r+1\end{matrix}\right) & \cdots\left(\begin{matrix}\overline{1\ldots n}\\2\ldots m\end{matrix}\right)\\ \cdots & \cdots & \cdots\ \cdots\\ \left(\begin{matrix}\overline{1\ldots n}\\m-r+1\ldots r\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\m-r+1\ldots r+1\end{matrix}\right) & \cdots\left(\begin{matrix}\overline{1\ldots n}\\m-r+1\ldots m\end{matrix}\right)\end{vmatrix}$$

$= say\ \Delta_{m-r+1}$.

Proof: It will be assumed that a relation shown in Theorem X to be valid for $\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m\end{matrix}\right)^{(m-1)}$ is valid for $\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m-1\end{matrix}\right)^{(r)}$, namely:

$$\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m-1\end{matrix}\right)^{(r)}=\begin{vmatrix}\left(\begin{matrix}\overline{1\ldots n}\\1\ldots r\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\1\ldots r+1\end{matrix}\right) & \cdots\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m-1\end{matrix}\right)\\ \left(\begin{matrix}\overline{1\ldots n}\\2\ldots r\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\2\ldots r+1\end{matrix}\right) & \cdots\left(\begin{matrix}\overline{1\ldots n}\\2\ldots m-1\end{matrix}\right)\\ \cdots & \cdots & \cdots\ \cdots\\ \left(\begin{matrix}\overline{1\ldots n}\\m-r\ldots r\end{matrix}\right) & \left(\begin{matrix}\overline{1\ldots n}\\m-r\ldots r+1\end{matrix}\right) & \cdots\left(\begin{matrix}\overline{1\ldots n}\\m-r\ldots m-1\end{matrix}\right)\end{vmatrix}$$

$=$ say Δ_{m-r}.

Then $\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m\end{matrix}\right)^{(r)}=\left(\begin{matrix}\overline{1\ldots n}\\m-r+1\ldots m\end{matrix}\right)\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m-1\end{matrix}\right)^{(r)}-\bar{X}_{m-r+1,\,m-1}$

$=(1-\phi)\left(\begin{matrix}\overline{1\ldots n}\\m-r+1\ldots m\end{matrix}\right)\left(\begin{matrix}\overline{1\ldots n}\\1\ldots m-1\end{matrix}\right)^{(r)}$,

by Theorem IX, corollary,

$=(1-\phi)\left(\begin{matrix}\overline{1\ldots n}\\m-r+1\ldots m\end{matrix}\right)\Delta_{m-r}=\Delta_{m-r+1}$, by Theorem V.

COROLLARY: $$\binom{\overline{1\ldots n}}{1\ldots m}^{(r)}=\begin{vmatrix} K'_{m-r+1} & K'_{m-r} & \ldots K'_1 \\ K'_{m-r+2} & K'_{m-r+1} & \ldots K'_2 \\ \ldots & \ldots & \ldots \\ K'_{2(m-r)+1} & K'_{2(m-r)} & \ldots K'_{m-r+1} \end{vmatrix}$$

where $K'_i=K_{n-m+i}, \quad m \ngtr n.$

Proof: $$\binom{\overline{1\ldots n}}{1\ldots m}^{(r)}=\begin{vmatrix} \underset{\alpha=1\ldots r}{K'_{m-r+1}} & \underset{\alpha=1\ldots r+1}{K'_{m-r}} & \underset{\alpha=1\ldots m}{K'_1} \\ \underset{\alpha=2\ldots r}{K'_{m-r+2}} & \underset{\alpha=2\ldots r+1}{K'_{m-r+1}} & \underset{\alpha=2\ldots m}{K'_2} \\ \ldots & \ldots & \ldots \\ \underset{\alpha=m-r+1\ldots r}{K'_{2(m-r)+1}} & \underset{\alpha=m-r+1\ldots r+1}{K'_{2(m-r)}} & \underset{\alpha=m-r+1\ldots m}{K'_{m-r+1}} \end{vmatrix},$$

by Theorems XII and I.

By means of Theorem IV the missing α's may be introduced as in the proof of the corollary of Theorem X, completing the proof.

THEOREM XIII. *For all values of* n,

$$\binom{\overline{1\ldots n}}{1\ldots m}^{(r)}=\begin{vmatrix} K_{n-r+1} & K_{n-r} & \ldots K_{n-m+1} \\ K_{n-r+2} & K_{n-r+1} & \ldots K_{n-m+2} \\ \ldots & \ldots & \ldots \\ K_{n-2r+m+1} & K_{n-2r+m} & \ldots K_{n-r+1} \end{vmatrix}. \qquad \text{(A)}$$

Proof: Theorem XII, corollary gives the result if $m \ngtr n$. If $m > n$, $\binom{\overline{1\ldots n}}{1\ldots m}^{(r)}$ can be calculated by the method of this paper with the interchange of rows and columns. The result will be the Δ' of Theorem VI where $t=m-r+1$ and $s=n-r+1$. But when $m>n$ the determinant of Theorem XII, corollary becomes the Δ of Theorem VI with the same values of t and s. The theorem follows.

COROLLARY: *For all values of* n,

$$\binom{\overline{1\ldots n}}{1\ldots m}^{(r)}=\begin{vmatrix} \binom{\overline{1\ldots n}}{1\ldots r} & \binom{\overline{1\ldots n}}{1\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{1\ldots m} \\ \binom{\overline{1\ldots n}}{2\ldots r} & \binom{\overline{1\ldots n}}{2\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{2\ldots m} \\ \ldots & \ldots & \ldots\;\ldots \\ \binom{\overline{1\ldots n}}{m-r+1\ldots r} & \binom{\overline{1\ldots n}}{m-r+1\ldots r+1} & \ldots\binom{\overline{1\ldots n}}{m-r+1\ldots m} \end{vmatrix}.$$

This follows from Theorem XIII by retracing the steps of the proof of Theorem XII, corollary.

Section 5.

THEOREM XIV. $\left(\overline{\begin{matrix}1\dots .n\\ 1\dots .m\end{matrix}}\right)^{(r)}$, *where each element* u_{ij} *is a function of degree* b *is* $\dfrac{\prod\limits_{i=0}^{i=m-r}{}_{n+m-r}C_{r-1+i}}{\prod\limits_{i=0}^{i=m-r}{}_{n+m-r}C_i}\cdot b^{(m-r+1)(n-r+1)}$

Proof: In this case the a's may be taken each equal to b, and the α's each equal to zero. K_i then reduces to $[\sum\limits_1^n a_1a_2\dots .a_i]_{a_j=b}={}_nC_rb^i$, and $\left(\overline{\begin{matrix}1\dots .n\\ 1\dots .m\end{matrix}}\right)^{(r)}$ to

$$b^{(m-r+1)(n-r+1)}\cdot\Delta \text{ where } \Delta=\begin{vmatrix}{}_nC_{n-r+1} & {}_nC_{n-r} & \dots . & {}_nC_{n-m+1}\\ {}_nC_{n-r+2} & {}_nC_{n-r+1} & \dots . & {}_nC_{n-m+2}\\ \dots\dots & \dots . & \dots . & \dots\dots\\ {}_nC_{n-2r+m+1} & {}_nC_{n-2r+m} & \dots . & {}_nC_{n-r+1}\end{vmatrix}.$$

Next we apply the formula ${}_nC_i={}_nC_{n-i}$ to each of the elements and multiply the resulting determinant by unity in the form of the determinant

$$\begin{vmatrix}{}_{m-r}C_0 & 0 & \dots . & 0\\ {}_{m-r}C_1 & {}_{m-r-1}C_0 & \dots . & 0\\ \dots . & \dots\dots & \dots . & .\\ {}_{m-r}C_{m-r} & {}_{m-r-1}C_{m-r-1} & \dots . & 1\end{vmatrix}.$$

This gives for the element in the h-th row and the k-th column,

$$\begin{aligned}{}_{n+m-r+1-h}C_{r-1-h+k}&=\frac{(n+m-r+1-h)!}{(r-1-h+k)!(n+m-2r+2-k)!}\\ &=\frac{(n+m-r+1-h)!\,{}_{r-2+k}P_{h-1}}{(r-2+k)!(n+m-2r+2-k)!},\ {}_lP_0=1.\end{aligned}$$

The factorial number in the numerator is constant for a given row as is the denominator for a given column. Hence $\Delta=A\Delta'$, where

$$A=\frac{\sum\limits_{h=1}^{h=m-r+1}(n+m-r+1-h)!}{\sum\limits_{k=1}^{k=m-r+1}(r-2+k)!(n+m-2r+2-k)!} \text{ and } \Delta'=\begin{vmatrix}h=1\dots .m-r+1\\ {}_{r-2+k}P_{h-1}\\ k=1\dots .m-r+1\end{vmatrix}=|a_{h,k}|.$$

Let $\underset{k=2\dots .m-r+1}{D'a_{h,k}}\equiv a_{h,k}-a_{h,k-1}$, $D'a_{h,1}=a_{h,1}$. Then $|D'a_{h,k}|\equiv\Delta'$.

$$\begin{aligned}D'a_{h,k}&={}_{r-2+k}P_{h-1}-{}_{r-3+k}P_{h-1}=[(r-2+k)-(r-1+k-h)]_{r-3+k}P_{h-2}\\ &=(h-1)_{r-3+k}P_{h-2},\end{aligned}$$

……

Finally

$$D^{(k-1)}a_{h,k}\equiv D^{(k-2)}a_{h,k}-D^{(k-2)}a_{h,k-1}=(h-1)(h-2)\ldots .(h-k+1)_{r-1}P_{h-k}$$
$$=\begin{cases}(h-1)!, \text{ if } k=h\\ 0, \text{ if } k>h.\end{cases}$$

Hence

$$\Delta'\equiv\begin{vmatrix}0! & 0 & 0 & 0 & \ldots . & 0\\ & 1! & 0 & 0 & \ldots . & 0\\ & & 2! & 0 & \ldots . & 0\\ .. & .. & .. & . & \ldots . & .\\ .. & .. & .. & . & \ldots . & 0\\ .. & .. & .. & . & \ldots . & (m-r)!\end{vmatrix}=\prod_{i=0}^{i=m-r} i!.$$

Hence $\Delta=A'\Delta=\dfrac{\prod\limits_{i=0}^{i=m-r}(n+m-r-i)!\prod\limits_{i=0}^{i=m-r}i!}{\prod\limits_{i=0}^{i=m-r}(r+i-1)!\prod\limits_{i=0}^{i=m-r}(n+m-2r+1-i)!}=\dfrac{\prod\limits_{i=0}^{i=m-r}{}_{n+m-r}C_{r-1+i}}{\prod\limits_{i=0}^{i=m-r}{}_{n+m-r}C_{i}}.$

The theorem follows.

COROLLARY: $\left(\begin{matrix}\overline{1\ldots . n}\\ 1\ldots . m\end{matrix}\right)^{(r)}$, *where each element* u_{ij} *is a function of degree* b, *is*

$$\frac{\prod\limits_{i=0}^{i=m-r}{}_{n+i}C_{n-r+1}}{\prod\limits_{i=0}^{i=m-r}{}_{n-r+1+i}C_{n-r+1}}\cdot b^{(m-r+1)(n-r+1)}.^{*} \qquad \text{(B)}$$

Proof: By changing the form of two of the factorial numbers in the value of $A\Delta'$, and introducing the factor $[(n-r+1)!]^{(m-r+1)}$ in the numerator and the denominator

$$A\Delta'=\frac{\prod\limits_{i=0}^{i=m-r}(n+i)!}{\prod\limits_{i=0}^{i=m-r}(r-1+i)!(n-r+1)!}\cdot\frac{\prod\limits_{i=0}^{i=m-r}i!(n-r+1)!}{\prod\limits_{i=0}^{i=m-r}(n-r+1+i)!}=\frac{\prod\limits_{i=0}^{i=m-r}{}_{n+i}C_{n-r+1}}{\prod\limits_{i=0}^{i=m-r}{}_{n-r+1+i}C_{n-r+1}}.$$

The corollary follows.

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[*] See paper by Segre, already referred to.